TRIGONOMETRIC SUBSTITUTIONS

We showed in class that the area enclosed by the circle $x^2 + y^2 = a^2$ equals πa^2. It involved computing integrals of the form

$$\int_0^a \sqrt{a^2 - x^2}\, dx$$

Exercise 1. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The following table is a list of trigonometric substitutions that we use for certain radical expressions.

Table 1. Table of Trigonometric Substitutions

Type	Expression	Substitution	Identity
1	$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
2	$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
3	$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Exercise 2. (Type 1) $\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

Note that the radical term is of Type 1 i.e. radical of the form $\sqrt{2^2 - x^2}$.
Therefore we need to use the substitution $x = 2\sin\theta$, and proceed from there.

Exercise 3. (Type 2) $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

Just because it has a radical expression does not necessarily mean that you need to use the substitution from the table. For instance:

Exercise 4. $\int \frac{x}{\sqrt{x^2+9}}\,dx$

Exercise 5. (Type 3) $\int \frac{1}{\sqrt{x^2-1}}\,dx$

So there are integrals where you need to do substitution first before computing the integrals

Exercise 6. $\int_0^{\frac{\sqrt{3}}{2}} \frac{x^3}{(4x^2+1)^{\frac{3}{2}}}\,dx$

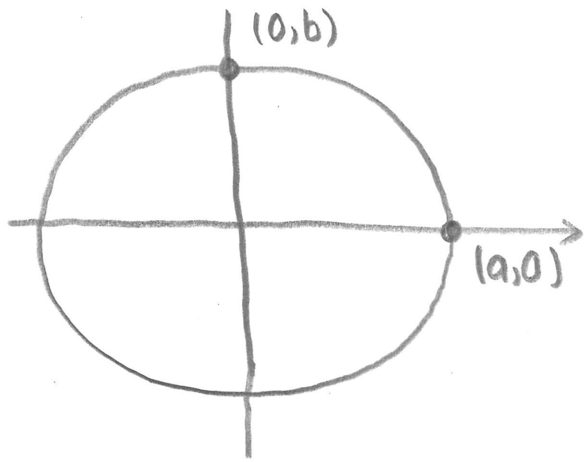
There are certain integrals where you need to solve by completing the squares first

Exercise 7. $\int \sqrt{5+4x-x^2}\,dx$

Exercise 8. $\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx$

TRIGONOMETRIC SUBSTITUTION

Ellipse : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



- Find the area enclosed by the ellipse

- All four areas are same, so we can restrict to the 1st quadrant.

Solving equation for y,

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = b^2\left(\frac{a^2-x^2}{a^2}\right)$$

$$\Rightarrow y^2 = \frac{b^2}{a^2}(a^2-x^2) \Rightarrow y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

Since $y > 0$, $y = \pm\frac{b}{a}\sqrt{a^2-x^2}$

Then,

$$A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

These exactly are the type of integrals that require trigonometric substitution.

We will set up a table but for time being,

$$x = a\sin\theta$$

$$\Rightarrow dx = a\cos\theta\,d\theta$$

<u>Limits of integration</u> $\quad 0 = a\sin\theta \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$\quad a = a\sin\theta \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$A = \int_0^{\pi/2} \frac{b}{a}\sqrt{a^2 - a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{b}{a} \sqrt{a^2(1-\sin^2\theta)}\; a\cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{b}{\cancel{a}} \cdot \cancel{a} \cdot \cos\theta \cdot a\cos\theta \, d\theta$$

$$= ab \int_0^{\pi/2} \cos^2\theta \, d\theta \quad = ab \int_0^{\pi/2} \left[\frac{1+\cos 2\theta}{2}\right] d\theta$$

$$= \frac{ab}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} \quad = \frac{ab}{2}\left[\frac{\pi}{2} + 0 - 0\right]$$

$$= \frac{\pi ab}{4}.$$

Then, total area $= \pi ab$.

Ex 2

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx$$

$x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta\,d\theta$

$$\sqrt{4-x^2} = \sqrt{4-(2\sin\theta)^2}$$
$$= \sqrt{4-4\sin^2\theta}$$
$$= \sqrt{4(1-\sin^2\theta)}$$
$$= 2\cos\theta$$

$$\int \frac{2\cos\theta}{(2\sin\theta)^2} \cdot 2\cos\theta\,d\theta$$

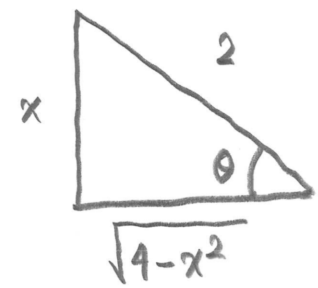
$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\,d\theta$$

$$= \int \operatorname{cosec}^2\theta - 1\,d\theta$$

$= -\cot\theta - \theta + C$

$= -\dfrac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\dfrac{x}{2}\right) + C$



3) $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

Type 2 $\quad a = 2$

$x = 2\tan\theta$

$\dfrac{dx}{d\theta} = 2\sec^2\theta \Rightarrow dx = 2\sec^2\theta\, d\theta$

$\sqrt{x^2+4} = \sqrt{(2\tan\theta)^2+4}$

$= \sqrt{4\tan^2\theta+4} = \sqrt{4(1+\tan^2\theta)} = 2\sec\theta$

$$= \int \frac{1}{(2\tan\theta)^2 \cdot 2\sec\theta} \cdot 2\sec^2\theta \, d\theta$$

$$= \frac{1}{4} \int \frac{\sec\theta}{\tan^2\theta} \, d\theta$$

$$= \frac{1}{4} \int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}} \, d\theta$$

$$= \frac{1}{4} \int \frac{\cos\theta}{\sin^2\theta} \, d\theta$$

let $v = \sin\theta$

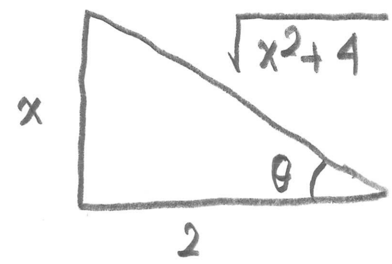
$dv = \cos\theta \, d\theta$

$$= \frac{1}{4} \int \frac{1}{v^2} \, dv$$

$$= \frac{1}{4} \frac{v^{-1}}{-1} + C$$

$$= \frac{-1}{4\sin\theta} + C$$

$$= -\frac{1}{4}\operatorname{cosec}\theta + C$$



$$= -\frac{1}{4}\,\frac{\sqrt{x^2+4}}{x} + C$$

4) $$\int \frac{x}{\sqrt{x^2+9}}\,dx$$

$$u = x^2+9 \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$$

$$\int \frac{x}{\sqrt{u}}\,\frac{du}{2x} = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\,2u^{1/2} + C$$

$$= u^{1/2} + C = \sqrt{x^2+9} + C$$

5) $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx$

Type 3 $\quad a = 1$

$x = \sec\theta$

$dx = \sec\theta \tan\theta \, d\theta$

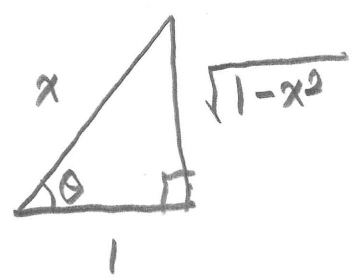
$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$$\int \frac{1}{\tan\theta} \sec\theta \tan\theta \, d\theta$$

$$= \int \sec\theta \, d\theta = \ln\left|\sec\theta + \tan\theta\right| + C$$

$\cos\theta = \dfrac{1}{x}$

$$= \ln\left|x + \sqrt{1-x^2}\right| + C$$



6) $\displaystyle\int_0^{\sqrt{3}/2} \frac{x^3}{(4x^2+1)^{3/2}}\,dx$

We want sth of the form u^2+1.

So $u^2 = 4x^2$ means

$u = 2x$

$du = 2dx \Rightarrow dx = \frac{du}{2}$

$x = 0,\ u = 0$

$x = \frac{\sqrt{3}}{2},\ u = \sqrt{3}$

$$= \int_0^{\sqrt{3}} \frac{(u/2)^3}{(u^2+1)^{3/2}}\,\frac{du}{2}$$

$$= \frac{1}{16}\int_0^{\sqrt{3}} \frac{u^3}{(u^2+1)^{3/2}}\,du$$

Type 2 $\quad a = 1,\ u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$(u^2+1)^{3/2} = (\tan^2\theta+1)^{3/2}$

$= \sec^3\theta$

$$= \frac{1}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec^3\theta}\sec^2\theta\,d\theta$$

$u = \sqrt{3} \qquad u = 0$

$\tan\theta = \sqrt{3} \qquad \tan\theta = 0$

$\theta = \frac{\pi}{3} \qquad \theta = 0$

$$= \frac{1}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$$= \frac{1}{16}\int_0^{\pi/3} \sec\theta \tan\theta \cdot \frac{\tan^2\theta}{\sec^2\theta} \, d\theta$$

$$= \frac{1}{16}\int_0^{\pi/3} \sec\theta \tan\theta \cdot \frac{\sec^2\theta - 1}{\sec^2\theta} \, d\theta$$

Let $u = \sec\theta$ $\qquad\qquad$ $\theta = 0,\ u = 1$

$\frac{du}{d\theta} = \sec\theta \tan\theta$ $\qquad\qquad$ $\theta = \frac{\pi}{3},\ u = 2$

$$= \frac{1}{16}\int_1^{2} \frac{u^2 - 1}{u^2} \, du$$

$$= \frac{1}{16}\int_1^{2} 1 - u^{-2} \, du$$

$$= \frac{1}{16}\left[u + u^{-1}\right]_1^2$$

$$= \frac{1}{16}\left[2 + \frac{1}{2} - (1+1)\right]$$

$$= \frac{1}{32}$$

7) $\int \sqrt{5+4x-x^2}\ dx$

Solve by completing square

$$5+4x-x^2 = -(x^2-4x-5)$$

$$= -\left(x^2-4x+\left(\frac{-4}{2}\right)^2-\left(\frac{-4}{2}\right)^2-5\right)$$

$$= -\left((x-2)^2-9\right)$$

$$= 9-(x-2)^2 = 3^2-(x-2)^2$$

$\int \sqrt{3^2-(x-2)^2}\ dx$ $\quad\quad u = x-2$

$du = dx$

$\int \sqrt{3^2-u^2}\ du$ $\quad\quad$ Type 1 $\quad a = 3$

$u = 3\sin\theta$

$du = 3\cos\theta\ d\theta$

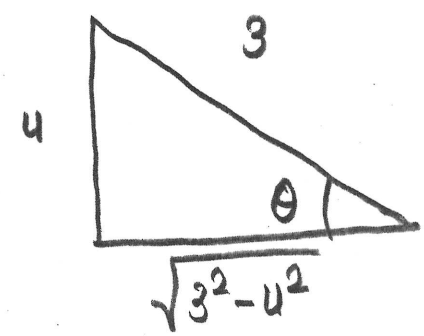
$$\sqrt{3^2-(3\sin\theta)^2} = \sqrt{9-9\sin^2\theta}$$

$$= \sqrt{9(1-\sin^2\theta)} = 3\cos\theta$$

$\nearrow$ even power $\cos\theta$

$$\int 3\cos\theta \cdot 3\cos\theta \, d\theta = 9\int \cos^2\theta \, d\theta = \frac{9}{2}\int 1+\cos 2\theta \, d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$



$$= \frac{9}{2}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] + C$$

$$= \frac{9}{2}\left(\sin^{-1}\left(\frac{u}{3}\right) + \frac{u}{3}\cdot\frac{\sqrt{3^2-u^2}}{3}\right) + C$$

$$= \frac{9}{2}\left[\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{x-2}{3}\cdot\frac{\sqrt{9-(x-2)^2}}{3}\right] + C$$